Note Title 6/6/2025

Talk3 - Emily - Opinion dynamics

Conventional model

$G = (V, E)$ a graph. (V: actors, E: communications)

$x \in \mathbb{R}^V$ a distribution of opinions on a single issue

Opinions evolve $\frac{dx}{dt} = -\alpha L x$ (Assumption)

L: graph Laplacian

$\rightsquigarrow$ Asymptotically stable equilibrium at consensus opinion

<u>Recap</u>: $\text{Graphs} \xrightarrow[\text{poset}]{\text{face}} \text{Posets} \longrightarrow \text{Spaces}$

$Q \longmapsto A(Q)$: point set Q, opens are upward closed subsets

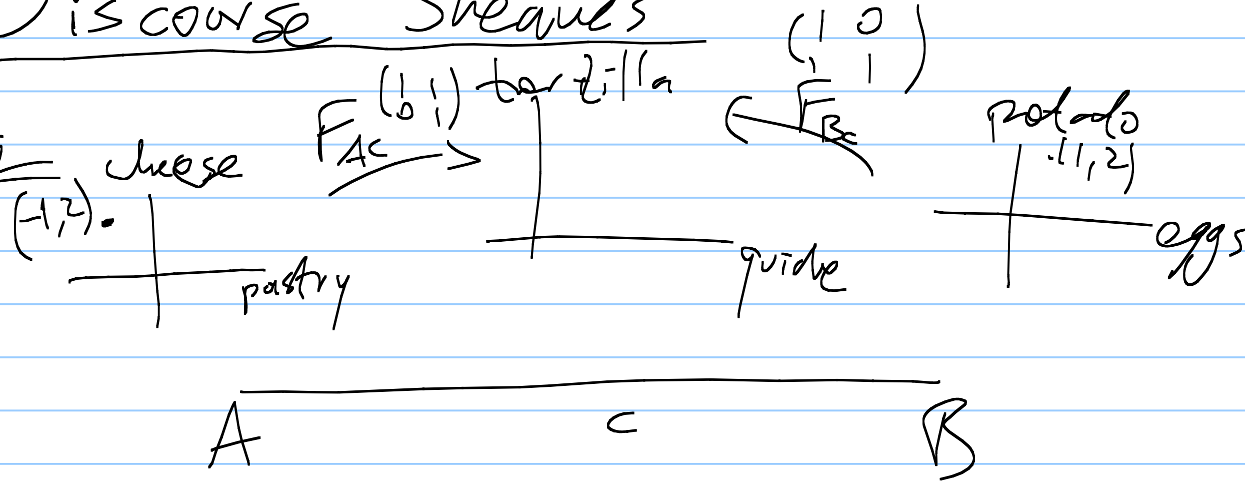
$G = (V, E) \longmapsto P(G) = (V \sqcup E, \trianglelefteq)$

A sheaf on a graph G is $F \in [P(G), \underline{\mathrm{Vect}}_{\mathbb{R}}]$

$\simeq \mathrm{Shv}(A(P(G)))$

Discourse Sheaves

Eg



tortilla = pastry, potato, eggs

quiche: cheese, pastry, eggs

Cohomology: Given $F \in [P(G), \underline{Vect}]$

- Choose orientation $\forall e \in E$
- $\underset{\bigoplus F(v)}{C^0(G;F)} \xrightarrow{\delta} \underset{\bigoplus F(e)}{C^1(G;F)}$, $(\delta x)_e = F_{v \trianglelefteq e}(x_v) - F_{u \trianglelefteq e}(x_u)$ if $u \leq v$

$H^0(G, F) = \ker(\delta) \rightsquigarrow$ (space of opinions w/ consensus?)

The sheaf Laplacian of F is $L_F = \delta^T \delta : C^0(G,F) \to C^0(G,F)$

$H^0(G,F) \cong \ker(L_F)$

$$(L_F x)_v = \sum_{u \overset{e}{—} v} F^T_{v \trianglelefteq e} \left(F_{v \trianglelefteq e} x_v - F_{u \trianglelefteq e} x_u \right)$$

<u>Ex</u>: Orient G as $A \xrightarrow{c} B$

Have $C^0(G,F) = \mathbb{R}^4 \xrightarrow{\delta} \mathbb{R}^2 = C^1(G,F)$

$\delta = \left[\begin{array}{cc:cc} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right]$ $\qquad H^0(G;F) = \langle (0,1,1,0), (-1,2,1,1) \rangle$

Diffusion of opinions: $\frac{dx}{dt} = -\alpha L_{\mathcal{F}}(x)$, $\alpha > 0$ $(\ast)$

$L_{\mathcal{F}}$ ← sheaf Laplacian

Thm: Solutions $x(t)$ to $(\ast)$ converge exp. as $t \to \infty$ to ~~the~~ orth. projection of x_0 onto $H^0(G; \mathcal{F})$.

asymptotically stable equilibrium at consensus opinion

Stubbornness:

$A \subset G$ a subgraph (a set of vertices & all edges), $\mathcal{F} \in [P(G), \underline{Vect}]$

[local sections over A]

$$C^0(A, \mathcal{F}) \xrightarrow{\delta} C^0(G, \mathcal{F})$$

$$C^0(G, A; \mathcal{F}) = \bigoplus_{v \in V \setminus A} \mathcal{F}(v) \xrightarrow{\delta} \bigoplus_{\substack{u \overset{e}{-} v \\ e \notin A\,?}} \mathcal{F}(e) = C^1(G, A; \mathcal{F})$$

$H^0(G, A; \mathcal{F}) =$ "cohomology rel. to A"

Def: Let $U \subset V(G)$ and $u \in C^0(U, \mathcal{F})$. A <u>harmonic extension</u> of u to G is $x \in C^0(G; \mathcal{F})$ such that $x|_U = u$ and $(L_{\mathcal{F}} x)_v = 0$ for all $v \in V(G) \setminus U$.

<u>**Thm:**</u> Every $u \in C^0(U, \mathcal{F})$ admits a harmonic extension. If $H^0(G, U; \mathcal{F}) = 0$ these are unique.

Thm: The dynamics $\frac{dx_v}{dt} = \begin{cases} -\alpha (L_F x)_v, & v \notin U \\ 0, & v \in U \end{cases}$, x_0 ← init cond.

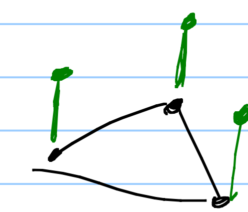
converge exp. to the harmonic extension of ~~x~~ closest in ℓ^2-norm to x_0.

Weighted reluctance

Fix a reluctance parameter γ_v for $v \in V(G)$

Let $\hat{G}$ have $V(\hat{G}) = V(G) \sqcup V(G) = V \sqcup V'$

$E(\hat{G}) = V(E) \sqcup \{(v,v')\} = E \sqcup E'$



Given discourse sheaf, $F \in [P(G), \underline{Vect}]$, Let $\hat{F}(v') = \hat{F}(\{v, v'\}) = F(v)$ and let $F_{v' \trianglelefteq e'} = F_{v \trianglelefteq e'} = \sqrt{\gamma_v}\, Id$.

Given $x_0 \in C^0(G, F)$, let $\hat{x}_0(v') = x_0(v)$

Make every $v' \in V'$ a stubborn agent.

Now
$$\frac{dx_v}{dt} = -\alpha (L_{\hat{F}} x)_v$$
$$= -\alpha (L_F x)_v + \alpha \gamma_v ((x_0)_v - x_v)$$